

# BOUNDEDNESS OF TRANSLATION OPERATOR ON VARIABLE LEBESGUE SPACES

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ABSTRACT. In this paper, we present more regularity conditions which ensure the boundedness of translation operator on variable Lebesgue spaces.

## 1. Introduction

Function spaces with variable exponents have been intensively studied in the recent years by a significant number of authors. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics [13], image restoration [2] and PDEs with non-standard growth conditions. Some example of these spaces can be mentioned such as: variable Lebesgue space, variable Besov and Triebel-Lizorkin spaces. We only refer to the papers [1, 4, 9-11] and to the monograph [7] for further details and references on recent developments on this field.

The purpose of the present paper is to study the translation operators  $\tau_h : f \mapsto f(\cdot + h)$ ,  $h \in \mathbb{R}^n$  in the framework of variable Lebesgue spaces  $L^{p(\cdot)}$ . Their behavior is well known if  $p$  is constant. In general  $\tau_h$  maps  $L^{p(\cdot)}$  to  $L^{p(\cdot)}$  for any  $h \in \mathbb{R}^n$  if and only if  $p$  is constant, see [5], Lemma 2.3. Allowing  $p$  to vary from point to point will raise extra difficulties which, in general, are overcome by imposing some regularity assumptions on this exponent.

As usual, we denote by  $\mathbb{R}^n$  the  $n$ -dimensional real Euclidean space,  $\mathbb{Z}$  is the set of all integer numbers,  $\mathbb{N}$  is the set of all natural numbers and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

We denote by  $B(x, r)$  the open ball in  $\mathbb{R}^n$  with center  $x$  and radius  $r$ . By  $\text{supp } f$  we denote the support of the function  $f$ , i.e., the closure of its non-zero set.

By  $\mathcal{S}(\mathbb{R}^n)$  we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on  $\mathbb{R}^n$  and by  $\mathcal{S}'(\mathbb{R}^n)$  the dual space of all tempered distributions on  $\mathbb{R}^n$ . We define the Fourier transform of a function  $f \in \mathcal{S}(\mathbb{R}^n)$  by  $\mathcal{F}(f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$ .

The Hardy-Littlewood maximal operator  $\mathcal{M}$  is defined on  $L^1_{\text{loc}}$  by

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

and  $M_{B(x+t, r)}f(x) := \frac{1}{|B(x+t, r)|} \int_{B(x+t, r)} |f(y)| dy$ ,  $t \in \mathbb{R}^n$  and  $r > 0$ . The variable exponents that we consider are always measurable functions on  $\mathbb{R}^n$  with range in  $[c, \infty)$  for

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some  $c > 0$ . We denote the set of such functions by  $\mathcal{P}_0$ . The subset of variable exponents with range  $[1, \infty)$  is denoted by  $\mathcal{P}$ . We use the standard notation  $p^- = \operatorname{ess-inf}_{x \in \mathbb{R}^n} p(x)$  and  $p^+ = \operatorname{ess-sup}_{x \in \mathbb{R}^n} p(x)$ .

The *variable exponent Lebesgue space*  $L^{p(\cdot)}$  is the class of all measurable functions  $f$  on  $\mathbb{R}^n$  such that the modular  $\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx$  is finite. This is a quasi-Banach function space equipped with the quasi-norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \mu > 0 : \varrho_{p(\cdot)}\left(\frac{1}{\mu}f\right) \leq 1 \right\}.$$

If  $p(x) := p$  is constant, then  $L^{p(\cdot)} = L^p$  is the classical Lebesgue space.

An useful property is that  $\varrho_{p(\cdot)}(f) \leq 1$  if and only if  $\|f\|_{p(\cdot)} \leq 1$  (*unit ball property*), which is clear for constant exponents since the relation between the norm and the modular is obvious in that case.

We say that a function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is *locally log-Hölder continuous*, if there exists a constant  $c_{\log} > 0$  such that

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\log(e + 1/|x - y|)}$$

for all  $x, y \in \mathbb{R}^n$ . If, for some  $g_{\infty} \in \mathbb{R}$  and  $c_{\log} > 0$ , there holds

$$|g(x) - g_{\infty}| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all  $x \in \mathbb{R}^n$ , then we say that  $g$  satisfies the *log-Hölder decay condition* (at infinity). Note that every function with log-decay condition is bounded.

The notation  $\mathcal{P}^{\log}$  is used for all those exponents  $p \in \mathcal{P}$  with  $\frac{1}{p}$  satisfies the local log-Hölder continuity condition and the log-Hölder decay condition, where we consider  $p_{\infty} := \lim_{|x| \rightarrow \infty} p(x)$ . The class  $\mathcal{P}_0^{\log}$  is defined analogously.

We refer to the recent monograph [7] for further details on all these properties, and historical remarks and references on variable exponent spaces. We also refer to the papers [3], [5], [8], [12] and [14] where various results on maximal function in variable Lebesgue spaces were obtained.

Recall that  $\eta_{v,m}(x) := 2^{nv} (1 + 2^v |x|)^{-m}$ , for any  $x \in \mathbb{R}^n$ ,  $v \in \mathbb{Z}$  and  $m > 0$ . Note that  $\eta_{v,m} \in L^1$  when  $m > n$  and that  $\|\eta_{v,m}\|_1 = c_m$  is independent of  $v$ .

By  $c$  we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g.  $c(p)$  means that  $c$  depends on  $p$ , etc.).

## 2. TECHNICAL LEMMAS

In this section we present some results which are useful for us. The next lemma often allows us to deal with exponents which are smaller than 1, see [6, Lemma A.7].

**Lemma 2.1.** *Let  $r > 0$ ,  $v \in \mathbb{N}_0$  and  $m > n$ . Then there exists  $c = c(r, m, n) > 0$  such that for all  $g \in \mathcal{S}'(\mathbb{R}^n)$  with  $\operatorname{supp} \mathcal{F}g \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{v+1}\}$ , we have*

$$|g(x)| \leq c(\eta_{v,m} * |g|^r(x))^{1/r}, \quad x \in \mathbb{R}^n.$$

We will make use of the following statement, can be proved by a similar arguments of [8], Lemma 3.3, see also [7], Theorem 3.2.4.

**Theorem 2.2.** Let  $p \in \mathcal{P}^{\log}$ ,  $\theta = 2 + \frac{|h|}{|Q|}$  and

$$M = \begin{cases} \exp(2 + \frac{|h|}{|Q|}) c_{\log}(p) / p^- & \text{if } |Q| < \min(|h|, 1) \\ 1 & \text{otherwise.} \end{cases}$$

Then for every  $m > 0$  there exists  $\gamma = \exp(-4mc_{\log}(1/p))$  such that

$$\begin{aligned} & \left( \frac{\gamma}{|Q|} \int_Q |f(y+h)| dy \right)^{p(x)} \\ & \leq M \int_Q |f(y+h)|^{p(y+h)} dy \\ & \quad + \min(1, |Q|^{\frac{m}{\theta}}) \left( (e + |x|)^{-m} + \frac{1}{|Q|} \int_Q (e + |y+h|)^{-m} dy \right) \end{aligned}$$

for every cube (or ball)  $Q \subset \mathbb{R}^n$ , all  $x \in Q$ ,  $h \in \mathbb{R}^n$  and all  $f \in L^{p(\cdot)} + L^\infty$  with  $\|f\|_{p(\cdot)} + \|f\|_\infty \leq 1$ .

Notice that if  $f \in L^{p(\cdot)} \cap L^\infty$  with  $\|f\|_{p(\cdot)} + \|f\|_\infty \leq 1$ , then we can move the term  $M$  and we can take  $\gamma = \exp(-mc_{\log}(1/p))$ .

The proof of this theorem is postponed to the Appendix.

### 3. TRANSLATION OPERATOR

Let  $p \in \mathcal{P}$  and define the translation operator by  $(\tau_h f)(y) := f(y+h)$ . It was shown in [5, Lemma 2.3], see also [7, Proposition 3.6.1.], that  $\tau_h$  maps  $L^{p(\cdot)}$  to  $L^{p(\cdot)}$  for any  $h \in \mathbb{R}^n$  if and only if  $p$  is constant. In this section we present regularity conditions, which ensure this operator is bounded on the spaces  $L^{p(\cdot)}$ .

**Theorem 3.1.** Let  $p \in \mathcal{P}^{\log}$  with  $1 < p^- \leq p^+ < \infty$  and  $h \in \mathbb{R}^n$ . Then for all  $f \in L^{p(\cdot)} \cap \mathcal{S}'(\mathbb{R}^n)$  with  $\text{supp } \mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{v+1}\}$ ,  $v \in \mathbb{N}_0$ , we have

$$\|\tau_h f\|_{p(\cdot)} \leq c \exp((2 + 2^{vn} |h|) c_{\log}(p)) \|f\|_{p(\cdot)},$$

where  $c > 0$  is independent of  $h$  and  $v$ .

*Proof.* Lemma 2.1 yields  $|f| \leq \eta_{v,N} * |f|$ , for any  $N > n, v \in \mathbb{N}_0$ . We write

$$\eta_{v,N} * |f|(x+h) = \int_{\mathbb{R}^n} \eta_{v,N}(x-y) |f(y+h)| dy.$$

We split the integral into two parts, one integral over the set  $B(x, 2^{-v})$  and one over its complement. The first part is bounded by

$$M_{B(x, 2^{-v})}(\tau_h f)(x),$$

and the second one is majorized by

$$c \sum_{i=0}^{\infty} 2^{(n-N)i} M_{B(x, 2^{1-v+i})}(\tau_h f)(x).$$

Consequently,

$$\|\tau_h f\|_{p(\cdot)} \leq c \sum_{i=0}^{\infty} 2^{(n-N)i} \|M_{B(\cdot, 2^{1-v+i})}(\tau_h f)\|_{p(\cdot)}.$$

We have

$$\left\| M_{B(\cdot, 2^{1-v+i})}(\tau_h f) \right\|_{p(\cdot)} = \|f\|_{p(\cdot)} \left\| M_{B(\cdot, 2^{1-v+i})} \left( \frac{\tau_h f}{\|f\|_{p(\cdot)}} \right) \right\|_{p(\cdot)}$$

and we will prove that

$$\left\| \gamma \lambda M_{B(\cdot, 2^{1-v+i})} \left( \frac{\tau_h f}{\|f\|_{p(\cdot)}} \right) \right\|_{p(\cdot)} \leq c, \quad i, v \in \mathbb{N}_0,$$

with  $c > 0$  independent of  $i, v$  and  $h$ ,  $\gamma = \exp(-4m c_{\log}(p))$  and

$$\lambda = \exp(-(2 + 2^{vn} |h|) c_{\log}(p)).$$

Taking into account Theorem 2.2 we have, for any  $i \in \mathbb{N}_0, m > 0$ ,

$$\left( \gamma \lambda 2^{(v-i-1)n} \int_{B(x, 2^{1-v+i})} \left| \frac{\tau_h f(y)}{\|f\|_{p(\cdot)}} \right| dy \right)^{p(x)/p^-} \quad (3.2)$$

we majorized it by, after a simple change of variable,

$$\begin{aligned} & M_{B(x+h, 2^{1-v+i})}(|g|^{p(\cdot)/p^-})(x) + (e + |x|)^{-m} \\ & + M_{B(x+h, 2^{1-v+i})}((e + |\cdot|)^{-m})(x), \end{aligned}$$

with  $g = \frac{f}{\|f\|_{p(\cdot)}}$ . Therefore the expression (3.2) is bounded by

$$c \mathcal{M}(|g|^{p(\cdot)/p^-})(x+h) + (e + |x|)^{-m} + c \mathcal{M}(e + |\cdot|^{-m})(x+h). \quad (3.3)$$

Obviously,

$$\begin{aligned} & \varrho_{p(\cdot)} \left( \gamma \lambda M_{B(\cdot, 2^{1-v+i})} \left( \frac{\tau_h f}{\|f\|_{p(\cdot)}} \right) \right) \\ & = 3^{p^-} \varrho_{p^-} \left( \frac{1}{3} \left( \gamma \lambda M_{B(\cdot, 2^{1-v+i})} \left( \frac{\tau_h f}{\|f\|_{p(\cdot)}} \right) \right)^{p(\cdot)/p^-} \right). \end{aligned}$$

In view of (3.3), the last term can be estimated by

$$\begin{aligned} & 3^{p^-} \left\| \mathcal{M}(|g|^{p(\cdot)/p^-})(\cdot + h) \right\|_{p^-}^{p^-} + 3^{p^-} \left\| (e + |\cdot|)^{-sm} \right\|_{p^-}^{p^-} \\ & + 3^{p^-} \left\| \mathcal{M}((e + |\cdot|)^{-m})(\cdot + h) \right\|_{p^-}^{p^-}. \end{aligned}$$

First we see that  $(e + |\cdot|)^{-sm} \in L^{p^-}$  for  $m > \frac{n+1}{sp^-}$ . Secondly the classical result on the continuity of  $\mathcal{M}$  on  $L^{p^-}$  implies that

$$\begin{aligned} \left\| \mathcal{M}(|g|^{p(\cdot)/p^-})(\cdot + h) \right\|_{p^-}^{p^-} & = \left\| \mathcal{M}(|g|^{p(\cdot)/p^-}) \right\|_{p^-}^{p^-} \\ & \leq c \left\| |g|^{p(\cdot)/p^-} \right\|_{p^-}^{p^-} \\ & = c \varrho_{p(\cdot)}(g) \\ & \leq c \end{aligned}$$

and

$$\left\| \mathcal{M}((e + |\cdot|)^{-m})(\cdot + h) \right\|_{p^-}^{p^-} = \left\| \mathcal{M}(e + |\cdot|)^{-m} \right\|_{p^-}^{p^-} \leq c \left\| (e + |\cdot|)^{-m} \right\|_{p^-}^{p^-} \leq c,$$

since  $m > \frac{n}{sp^-}$  (with  $c > 0$  independent of  $h$ ). Hence there exists a constant  $C > 0$  independent of  $i$  and  $v$  such that

$$\varrho_{p(\cdot)}\left(\gamma\lambda M_{B(\cdot, 2^{1-v+i})}\left(\frac{\tau_h f}{\|f\|_{p(\cdot)}}\right)\right) \leq C.$$

Consequently, we have for any  $N > n$

$$\begin{aligned} \|\tau_h f\|_{p(\cdot)} &\leq C\lambda \sum_{i=0}^{\infty} 2^{(n-N)i} \|f\|_{p(\cdot)} \\ &\leq c \exp((2 + 2^{vn} |h|) c_{\log}(p)) \|f\|_{p(\cdot)}. \end{aligned}$$

The proof is complete.  $\square$

*Remark 3.4.* (i) Using Lemma 2.1 we can extend this theorem to the case of  $p \in \mathcal{P}_0^{\log}$  with  $0 < p^- \leq p^+ < \infty$ .

(ii) Let  $p \in \mathcal{P}^{\log}$  with  $1 < p^- \leq p^+ < \infty$  and  $f \in L^{p(\cdot)}$  with  $\text{supp } \mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{v+1}\}$ ,  $v \in \mathbb{N}_0$ . From Theorem 3.1, we easily obtain

$$\|f * g\|_{p(\cdot)} \leq c \|\exp((2 + 2^{vn} |\cdot|) c_{\log}(p))g\|_1 \|f\|_{p(\cdot)},$$

provided that the first norm on the right-hand side is finite, where  $c > 0$  is independent of  $v$ .

#### 4. APPENDIX

In this appendix we present the proof of Theorem 2.2. Our estimate use partially some decomposition techniques already used in [8, Lemma 3.3], see also [7, Theorem 4.2.4]. Let  $p \in \mathcal{P}^{\log}$  with  $1 \leq p^- \leq p^+ < \infty$  and  $p_{Q+h}^- = \text{ess-inf}_{z \in Q} p(z+h)$ . Define  $q \in \mathcal{P}^{\log}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$  by

$$\frac{1}{q(x, y, h)} = \max\left(\frac{1}{p(x)} - \frac{1}{p(y+h)}, 0\right).$$

Then

$$\begin{aligned} &\left(\frac{\gamma}{|Q|} \int_Q |f(y+h)| dy\right)^{p(x)} \\ &\leq \frac{M}{|Q|} \int_Q |f(y+h)|^{p(y+h)} dy + \frac{1}{|Q|} \int_Q \gamma^{q(x,y,h)} dy, \end{aligned}$$

for every cube  $Q \subset \mathbb{R}^n$ , all  $x \in Q$ ,  $h \in \mathbb{R}^n$  and all  $f \in L^{p(\cdot)} + L^\infty$  with  $\|f\|_{p(\cdot)} + \|f\|_\infty \leq 1$ . Indeed, we split  $f(y+h)$  into three parts

$$\begin{aligned} f_1(y+h) &= f(y+h) \chi_{\{y: |f(y+h)| > 1\}}(y) \\ f_2(y+h) &= f(y+h) \chi_{\{y: |f(y+h)| \leq 1, p(y+h) \leq p(x)\}}(y) \\ f_3(y+h) &= f(y+h) \chi_{\{y: |f(y+h)| \leq 1, p(y+h) > p(x)\}}(y), \end{aligned}$$

By convexity of  $t \mapsto t^p$ ,

$$\begin{aligned} \left(\frac{\gamma}{|Q|} \int_Q |f(y+h)| dy\right)^{p(x)} &\leq 3^{p^+-1} \sum_{i=1}^3 \left(\frac{\gamma}{|Q|} \int_Q |f_i(y+h)| dy\right)^{p(x)} \\ &= 3^{p^+-1} (I_1 + I_2 + I_3). \end{aligned}$$

**Estimation of  $I_1$ .** We divide the estimation in three cases.

**Case 1.**  $p(x) \leq p_{Q+h}^-$ . By Jensen's inequality,

$$I_1 \leq \gamma^{p(x)} \frac{1}{|Q|} \int_Q |f_1(y+h)|^{p(x)} dy = I.$$

Since  $|f_1(y+h)| > 1$ , we have  $|f_1(y+h)|^{p(x)} \leq |f_1(y+h)|^{p(y+h)}$  and thus

$$I \leq \frac{1}{|Q|} \int_Q |f(y+h)|^{p(y+h)} dy.$$

If  $\|f\|_\infty \leq 1$ , then  $f_1(y+h) = 0$  and  $I = 0$ .

**Case 2.**  $p(x) > p_{Q+h}^- \geq p_Q^-$ . Again Jensen's inequality implies that

$$\begin{aligned} I_1 &\leq \left( \frac{\gamma}{|Q|} \int_Q |f_1(y+h)|^{p_Q^-} dy \right)^{\frac{p(x)}{p_Q^-}} \\ &\leq \left( \frac{\gamma}{|Q|} \int_Q |f(y+h)|^{p(y+h)} dy \right)^{\frac{p(x)}{p_Q^-} - 1} \left( \frac{\gamma}{|Q|} \int_Q |f(y+h)|^{p(y+h)} dy \right) \\ &\leq c \frac{\gamma}{|Q|} \int_Q |f(y+h)|^{p(y+h)} dy, \end{aligned}$$

by the fact that  $\int_Q |f(y+h)|^{p(y+h)} dy \leq 1$  and  $\left(\frac{1}{|Q|}\right)^{\frac{p(x)}{p_Q^-} - 1} \leq c$ , which follow from  $p \in \mathcal{P}^{\log}$ , with  $c > 0$  independent of  $x, h$  and  $|Q|$ .

**Case 3.**  $p(x) \geq p_Q^- > p_{Q+h}^-$ . We have

$$\begin{aligned} I_1 &\leq \left( \frac{\gamma}{|Q|} \int_Q |f_1(y+h)|^{p_{Q+h}^-} dy \right)^{\frac{p(x)}{p_{Q+h}^-}} \\ &\leq \left( \frac{\gamma}{|Q|} \int_Q |f(y+h)|^{p(y+h)} dy \right) \left( \frac{\gamma}{|Q|} \int_Q |f(y+h)|^{p(y+h)} dy \right)^{\frac{p(x)}{p_{Q+h}^-} - 1} \\ &\leq \frac{1}{|Q|} \int_Q |f(y+h)|^{p(y+h)} dy \left( \frac{1}{|Q|} \right)^{\frac{p(x)}{p_{Q+h}^-} - 1}. \end{aligned}$$

If  $|Q| \geq 1$ , then the second term is bounded by 1. Now we suppose that  $|Q| < 1$ . We use the local log-Hölder condition:

$$\left( \frac{1}{|Q|} \right)^{\frac{p(x)}{p_{Q+h}^-} - 1} = \left( \frac{1}{|Q|} \right)^{\frac{p(x) - p_Q^-}{p_{Q+h}^-}} \left( \frac{1}{|Q|} \right)^{\frac{p_Q^- - p_{Q+h}^-}{p_{Q+h}^-}} \leq c \left( \frac{1}{|Q|} \right)^{\frac{p_Q^- - p_{Q+h}^-}{p_{Q+h}^-}}.$$

Let  $p(x_0) = p_Q^-$  and  $p(y_0+h) = p_{Q+h}^-$  with  $x_0, y_0 \in Q$ . Since  $p \in \mathcal{P}^{\log}$ , we have

$$\begin{aligned} \left( \frac{1}{|Q|} \right)^{\frac{p_Q^- - p_{Q+h}^-}{p_{Q+h}^-}} &= \left( \frac{1}{|Q|} \right)^{\frac{p(x_0) - p(y_0)}{p_{Q+h}^-}} \left( \frac{1}{|Q|} \right)^{\frac{p(y_0) - p(y_0+h)}{p_{Q+h}^-}} \\ &\leq c \left( \frac{1}{|Q|} \right)^{\frac{p(y_0) - p(y_0+h)}{p_{Q+h}^-}}. \end{aligned}$$

We see that

$$|p(y_0) - p(y_0 + h)| \leq \sum_{i=0}^{N-1} \left| p(y_0 + \frac{i}{N}h) - p(y_0 + \frac{i+1}{N}h) \right|,$$

where

$$N = \begin{cases} \left[ \frac{|h|}{|Q|} \right] + 1 & \text{if } |h| > |Q| \\ 1 & \text{otherwise.} \end{cases}$$

Therefore,

$$\left( \frac{1}{|Q|} \right)^{\frac{p(y_0) - p(y_0 + h)}{p_{\bar{Q}+h}}} \leq \prod_{i=0}^{N-1} \left( \frac{1}{|Q|} \right)^{\frac{p(y_0 + \frac{i}{N}h) - p(y_0 + \frac{i+1}{N}h)}{p_{\bar{Q}+h}}} \leq c \exp(N c_{\log}(p) / p^-),$$

where  $c > 0$  independent of  $y_0, h, N$  and  $|Q|$ .

**Estimation of  $I_2$ .** By Jensen's inequality,

$$I_2 \leq \gamma^{p(x)} \frac{1}{|Q|} \int_Q |f_2(y + h)|^{p(x)} dy = J.$$

Since  $|f_2(y + h)| \leq 1$  we have  $|f_2(y + h)|^{p(x)} \leq |f_2(y + h)|^{p(y+h)}$  and thus

$$J \leq \frac{1}{|Q|} \int_Q |f(y + h)|^{p(y+h)} dy.$$

**Estimation of  $I_3$ .** Again by Jensen's inequality,

$$\begin{aligned} & \left( \frac{\gamma}{|Q|} \int_Q |f_3(y + h)| dy \right)^{p(x)} \\ & \leq \frac{1}{|Q|} \int_Q (|\gamma f(y + h)|)^{p(x)} \chi_{\{|f(y+h)| \leq 1, p(y+h) > p(x)\}}(y) dy. \end{aligned}$$

Now, Young's inequality give that the last term is bounded by

$$\frac{1}{|Q|} \int_Q \left( |f(y + h)|^{p(y+h)} + \gamma^{q(x,y,h)} \right) dy.$$

Observe that

$$\frac{1}{q(x, y, h)} = \max\left(\frac{1}{p(x)} - \frac{1}{p(y+h)}, 0\right) \leq \frac{1}{s(x)} + \frac{1}{s(y+h)},$$

where  $\frac{1}{s(\cdot)} = \left| \frac{1}{p(\cdot)} - \frac{1}{p_\infty} \right|$ . We have

$$\gamma^{q(x,y,h)} = \gamma^{q(x,y,h)/2} \gamma^{q(x,y,h)/2} \leq \gamma^{q(x,y,h)/2} \left( \gamma^{s(x)/4} + \gamma^{s(y+h)/4} \right).$$

We suppose that  $|Q| < 1$ . Then

$$\begin{aligned} \left| \frac{1}{q(x, y, h)} \right| & \leq \left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| + \left| \frac{1}{p(y)} - \frac{1}{p(y+h)} \right| \\ & \leq \frac{c_{\log}(1/p)}{-\log|Q|} + \sum_{i=0}^{N-1} \left| \frac{1}{p(y + \frac{i}{N}h)} - \frac{1}{p(y + \frac{i+1}{N}h)} \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \frac{1}{q(x, y, h)} \right| &\leq \frac{c_{\log}(1/p)}{-\log|Q|} + \sum_{i=0}^{N-1} \frac{c_{\log}(1/p)}{\log\left(e + \frac{N}{|h|}\right)} \leq \frac{c_{\log}(1/p)}{-\log|Q|} (1 + N) \\ &\leq \frac{c_{\log}(1/p)}{-\log|Q|} \left(2 + \frac{|h|}{|Q|}\right). \end{aligned}$$

Hence,  $\gamma^{q(x,y,h)/2} = \gamma^{\frac{q(x,y,h)}{4}} \gamma^{\frac{q(x,y,h)}{4}} \leq |Q|^{\frac{m}{2+\frac{|h|}{|Q|}}} \gamma^{\frac{q(x,y,h)}{4}}$ . If  $|Q| \geq 1$ , then we use  $\gamma^{q(x,y,h)/2} \leq 1$  which follow from  $\gamma < 1$ . Now by [7, Proposition 4.1.8], see also [8, Lemma 3.3], we obtain the desired inequality.

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